

# ABELIAN HERMITIAN GEOMETRY

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**ABSTRACT.** We study the structure of Lie groups admitting left invariant abelian complex structures in terms of commutative associative algebras. If, in addition, the Lie group is equipped with a left invariant Hermitian structure, it turns out that such a Hermitian structure is Kähler if and only if the Lie group is the direct product of several copies of the real hyperbolic plane by a euclidean factor. Moreover, we show that if a left invariant Hermitian metric on a Lie group with an abelian complex structure has flat first canonical connection, then the Lie group is abelian.

## 1. INTRODUCTION

An *abelian* complex structure on a real Lie algebra  $\mathfrak{g}$  is an endomorphism  $J$  of  $\mathfrak{g}$  satisfying

$$J^2 = -I, \quad [Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g},$$

or equivalently, the  $i$ -eigenspace of  $J$  in the complexification  $\mathfrak{g}^{\mathbb{C}}$  of  $\mathfrak{g}$  is an abelian subalgebra of  $\mathfrak{g}^{\mathbb{C}}$ . If  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$  these conditions imply the vanishing of the Nijenhuis tensor on the invariant almost complex manifold  $(G, J)$ , that is,  $J$  is integrable on  $G$ . If  $\Gamma \subset G$  is any discrete subgroup of  $G$  then the induced  $J$  on  $\Gamma \backslash G$  will be also called invariant.

Our interest arises from properties of the complex manifolds obtained by considering this class of complex structures on Lie algebras. For instance, an abelian hypercomplex structure on  $\mathfrak{g}$ , that is, a pair of anticommuting abelian complex structures, gives rise to an invariant weak HKT structure on  $G$  (see [16] and [18]). It was proved in [10] that the converse of this result holds for nilmanifolds, that is, the hypercomplex structure associated to a nilmanifold carrying an HKT structure is necessarily abelian. Also, in [12, 13, 21] it was shown that invariant abelian complex structures on nilmanifolds have a locally complete family of deformations consisting of invariant complex structures.

In the first part of the article, we consider a distinguished class of Lie algebras admitting abelian complex structures, given by abelian double products. An abelian double product is a Lie algebra  $\mathfrak{g}$  together with an abelian complex structure  $J$  and a decomposition  $\mathfrak{g} = \mathfrak{u} \oplus J\mathfrak{u}$ , where both,  $\mathfrak{u}$  and  $J\mathfrak{u}$  are abelian subalgebras. The structure of these Lie algebras can be described in terms of a pair of commutative associative algebras satisfying a compatibility condition (see [5, 6]). When  $\mathfrak{u}$  is an ideal of  $\mathfrak{g}$  one obtains  $\text{aff}(\mathcal{A})$ , where  $\mathcal{A}$  is a commutative associative algebra, a class of Lie algebras considered in [8]. On the other hand, when  $\mathfrak{g}$  is a Lie algebra with an abelian complex structure  $J$ , we prove in Theorem 3.5 that if  $\mathfrak{g}$  decomposes as  $\mathfrak{g} = \mathfrak{u} + J\mathfrak{u}$ , with  $\mathfrak{u}$  an abelian subalgebra, then  $\mathfrak{g}$  is an abelian double product. Moreover, if  $\mathfrak{u}$  is an abelian ideal and  $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$ , then  $\mathfrak{g}$  is an affine Lie algebra  $\text{aff}(\mathcal{A})$ , for some commutative associative algebra  $\mathcal{A}$ .

In the second part, the Hermitian geometry of Lie groups equipped with abelian complex structures is studied. First, and related with results mentioned above, we obtain in Theorem 4.1 and Corollary 4.2 that such a Lie group satisfying the Kähler condition is the product of hyperbolic

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spaces and a flat factor. Second, we investigate the first canonical Hermitian connection on Lie groups associated to a left invariant Hermitian structure with abelian complex structure. This connection, which was introduced by Lichnerowicz in [20] and appears in the set of canonical Hermitian connections of Gauduchon [17], has torsion tensor of type  $(1, 1)$  with respect to the complex structure. In [4], the notion of abelian complex structure on Lie groups was extended to parallelizable manifolds. This generalization amounts to the existence of a complex connection on the complex manifold with trivial holonomy and torsion of type  $(1, 1)$  with respect to the complex structure. This motivates our study of the flatness of the first canonical Hermitian connection in the abelian case. We prove that if the first canonical connection associated to a left invariant Hermitian structure with abelian complex structure is flat, then the Lie group is abelian.

## 2. PRELIMINARIES

In this section we will recall basic definitions and known results which will be used throughout the paper.

**2.1. Connections on Lie algebras.** Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$  and suppose that  $G$  admits a left invariant affine connection  $\nabla$ , i.e., each left translation is an affine transformation of  $G$ . In this case, if  $X, Y$  are two left invariant vector fields on  $G$  then  $\nabla_X Y$  is also left invariant. Moreover, there is a one-to-one correspondence between the set of left invariant connections on  $G$  and the set of  $\mathfrak{g}$ -valued bilinear forms  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  (see [19, p.102]). Therefore, such a bilinear form  $\nabla : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  will be called a connection on  $\mathfrak{g}$ . The torsion  $T$  and the curvature of  $\nabla$  are defined as follows:

$$\begin{aligned} T(x, y) &= \nabla_x y - \nabla_y x - [x, y] \\ R(x, y) &= \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]}, \end{aligned}$$

for any  $x, y \in \mathfrak{g}$ . The connection  $\nabla$  is torsion-free when  $T = 0$ , and  $\nabla$  is flat when  $R = 0$ . We note that in the flat case,  $\nabla : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  is a representation of  $\mathfrak{g}$ .

The connection  $\nabla$  on  $\mathfrak{g}$  defined by  $\nabla = 0$  is known as the  $(-)$ -connection. Its torsion  $T$  is given by  $T(x, y) = -[x, y]$  for all  $x, y \in \mathfrak{g}$  and clearly  $R = 0$ .

**2.2. Complex structures on Lie algebras.** A complex structure on a Lie algebra  $\mathfrak{g}$  is an endomorphism  $J$  of  $\mathfrak{g}$  satisfying  $J^2 = -I$  together with the vanishing of the Nijenhuis bilinear form with values in  $\mathfrak{g}$ ,

$$(1) \quad N(x, y) = [Jx, Jy] - J[Jx, y] - J[x, Jy] - [x, y],$$

where  $x, y \in \mathfrak{g}$ .

If  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$ , by left translating the endomorphism  $J$  we obtain a complex manifold  $(G, J)$  such that left translations are holomorphic maps. A complex structure of this kind is called left invariant. We point out that  $(G, J)$  is not necessarily a complex Lie group since right translations are not in general holomorphic.

Consider a Lie algebra  $\mathfrak{g}$  equipped with a complex structure  $J$ . We denote with  $\mathfrak{g}'$  the commutator ideal  $[\mathfrak{g}, \mathfrak{g}]$  of  $\mathfrak{g}$  and with  $\mathfrak{g}'_J$  the  $J$ -stable ideal of  $\mathfrak{g}$  defined by  $\mathfrak{g}'_J := \mathfrak{g}' + J\mathfrak{g}'$ .

We will consider connections on  $\mathfrak{g}$  compatible with the complex structure  $J$ , and therefore we will say that a connection  $\nabla$  on  $\mathfrak{g}$  is complex when  $\nabla J = 0$ , that is,  $\nabla_x J = J\nabla_x$  for any  $x \in \mathfrak{g}$ .

The torsion  $T$  of a connection  $\nabla$  on  $\mathfrak{g}$  is said to be of type  $(1, 1)$  if  $T(Jx, Jy) = T(x, y)$  for all  $x, y$  on  $\mathfrak{g}$ .

**2.3. Abelian complex structures.** An *abelian* complex structure on  $\mathfrak{g}$  is an endomorphism  $J$  of  $\mathfrak{g}$  satisfying

$$(2) \quad J^2 = -I, \quad [Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g}.$$

It follows that condition (2) is a particular case of (1). We note that the  $(-)$ -connection on  $\mathfrak{g}$  has torsion of type  $(1, 1)$  if and only if  $J$  is abelian.

Abelian complex structures have been studied by several authors [13, 14, 21, 25]. A complete classification of the Lie algebras admitting abelian complex structures is known up to dimension 6 (see [3]) and there are structure results for arbitrary dimensions (see [8]).

The next result states some properties of Lie algebras admitting abelian complex structures that will be used in forthcoming sections.

**Lemma 2.1.** *Let  $J$  be an abelian complex structure on the Lie algebra  $\mathfrak{g}$ . Then:*

- (i) *the center  $\mathfrak{z}$  of  $\mathfrak{g}$  is  $J$ -stable;*
- (ii) *for any  $x \in \mathfrak{g}$ ,  $\text{ad}_{Jx} = -\text{ad}_x J$ ;*
- (iii)  *$\mathfrak{g}'$  is abelian, equivalently,  $\mathfrak{g}$  is 2-step solvable;*
- (iv)  *$J\mathfrak{g}'$  is an abelian subalgebra;*
- (v)  *$\mathfrak{g}' \cap J\mathfrak{g}' \subseteq \mathfrak{z}(\mathfrak{g}_J)$ .*

*Proof.* (i) and (ii) are straightforward. (iii) is a consequence of results in [23]. Using (iii) and the fact that  $J$  is abelian, (iv) follows.

If  $x \in \mathfrak{g}' \cap J\mathfrak{g}'$ , then (iii) and (iv) imply that  $[x, \mathfrak{g}'] = 0 = [x, J\mathfrak{g}']$ , thus  $x \in \mathfrak{z}(\mathfrak{g}_J)$  and (v) holds.  $\square$

**2.4. Hermitian Lie algebras.** Let  $\mathfrak{g}$  be a Lie algebra endowed with an inner product  $g$ . A connection  $\nabla$  on  $\mathfrak{g}$  is called metric if  $\nabla g = 0$ , that is,  $\nabla_x$  is a skew-symmetric endomorphism of  $\mathfrak{g}$  for any  $x \in \mathfrak{g}$ . When  $\mathfrak{g}$  is solvable and the connection  $\nabla$  is metric and flat, we have the following consequence:

**Lemma 2.2.** *Let  $\mathfrak{g}$  be a solvable Lie algebra equipped with an inner product  $g$  and a flat metric connection  $\nabla$ . Then  $\{\nabla_x : x \in \mathfrak{g}\}$  is a commutative family of skew-symmetric endomorphisms of  $\mathfrak{g}$  and  $\nabla_v = 0$  for every  $v \in \mathfrak{g}'$ .*

*Proof.* Since  $\nabla$  is flat it defines a representation of  $\mathfrak{g}$  on itself, hence the image of  $\nabla$  is a solvable Lie subalgebra of  $\mathfrak{gl}(\mathfrak{g})$ . Moreover, since  $\nabla g = 0$ , this solvable Lie subalgebra is contained in  $\mathfrak{so}(\mathfrak{g})$ , therefore it is abelian. Now  $\nabla_{[x,y]} = [\nabla_x, \nabla_y] = 0$ , and the lemma follows.  $\square$

The Levi-Civita connection  $\nabla^g$  associated to  $g$  is the only torsion-free metric connection on  $\mathfrak{g}$ , and it is given by

$$(3) \quad 2g(\nabla_x^g y, z) = g([x, y], z) - g([y, z], x) + g([z, x], y), \quad x, y, z \in \mathfrak{g}.$$

If  $J$  is a complex structure on  $\mathfrak{g}$  compatible with  $g$ , that is,  $g(x, y) = g(Jx, Jy)$  for all  $x, y \in \mathfrak{g}$ , then  $(J, g)$  is called a Hermitian structure on  $\mathfrak{g}$  and the triple  $(\mathfrak{g}, J, g)$  is a Hermitian Lie algebra. The associated Kähler form  $\omega$  is defined by  $\omega(x, y) = g(Jx, y)$ . If  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$ , by left-translating  $J$  and the inner product  $g$  we give to  $G$  a left invariant Hermitian structure. A Hermitian Lie algebra  $(\mathfrak{g}, J, g)$  is called *Kähler* if  $\nabla^g$  is a complex connection or, equivalently,  $d\omega = 0$ , where

$$(4) \quad d\omega(x, y, z) = -\omega([x, y], z) - \omega([y, z], x) - \omega([z, x], y), \quad x, y, z \in \mathfrak{g}.$$

### 3. ABELIAN DOUBLE PRODUCTS AND AFFINE LIE ALGEBRAS

In this section we consider a particular class of Lie algebras equipped with abelian complex structures, the so called abelian double products, and as a consequence of the main result of this section (Theorem 3.5) we obtain that  $\mathfrak{g}'_J$  belongs to this class, whenever  $\mathfrak{g}$  is a Lie algebra and  $J$  an abelian complex structure.

A *complex product structure* on  $\mathfrak{g}$  is a complex structure  $J$  together with a decomposition  $\mathfrak{g} = \mathfrak{u} \oplus J\mathfrak{u}$ , where both,  $\mathfrak{u}$  and  $J\mathfrak{u}$ , are Lie subalgebras of  $\mathfrak{g}$ . If  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$ , the subalgebras  $\mathfrak{u}$  and  $J\mathfrak{u}$  give rise to involutive distributions  $TG^+$  and  $TG^-$ , respectively, such that  $TG = TG^+ \oplus TG^-$  and  $J(TG^+) = TG^-$ , where  $J$  is the induced left invariant complex structure on  $G$ . We obtain in this way a left invariant complex product structure on  $G$  (see [6]).

It was proved in [5] that the Lie subalgebras  $\mathfrak{u}$  and  $J\mathfrak{u}$  are abelian if and only if the complex structure  $J$  is abelian. In this case we will say that  $\mathfrak{g}$  is an *abelian double product*.

**Example 1.** The Lie algebra of the affine motion group of  $\mathbb{C}$ , denoted  $\mathfrak{aff}(\mathbb{C})$ , is an abelian double product with respect to two non-equivalent abelian complex structures. Indeed,  $\mathfrak{aff}(\mathbb{C})$  has a basis  $\{e_1, e_2, e_3, e_4\}$  with the following Lie bracket

$$(5) \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_2, e_3] = e_4, \quad [e_2, e_4] = -e_3.$$

Any abelian complex structure on  $\mathfrak{aff}(\mathbb{C})$  is equivalent to one and only one of the following (see [3, Proposition 2.6]):

$$J_1 e_1 = -e_2, \quad J_1 e_3 = e_4, \quad \text{or} \quad J_2 e_1 = e_3, \quad J_2 e_2 = e_4.$$

It follows that  $\mathfrak{aff}(\mathbb{C})$  is an abelian double product with respect to both  $J_1$  and  $J_2$ , by setting  $\mathfrak{u}_1 = \text{span}\{e_1 + e_3, e_2 + e_4\}$  and  $\mathfrak{u}_2 = \text{span}\{e_1, e_2\}$ , respectively.

We introduce next a family of Lie algebras which exhausts the class of abelian double products. They are obtained by considering a finite dimensional real vector space  $\mathcal{A}$  with two structures of commutative associative algebra,  $(\mathcal{A}, \cdot)$  and  $(\mathcal{A}, *)$ , such that both products satisfy the compatibility conditions

$$(6) \quad a * (b \cdot c) = b * (a \cdot c), \quad a \cdot (b * c) = b \cdot (a * c),$$

for every  $a, b, c \in \mathcal{A}$ . Then,  $\mathcal{A} \oplus \mathcal{A}$  with the bracket:

$$(7) \quad [(a, a'), (b, b')] = (-(a * b' - b * a'), a \cdot b' - b \cdot a'), \quad a, b, a', b' \in \mathcal{A},$$

and the endomorphism  $J$  defined by

$$(8) \quad J(a, a') = (-a', a), \quad a, a' \in \mathcal{A},$$

is an abelian double product that will be denoted  $(\mathcal{A}, \cdot) \bowtie (\mathcal{A}, *)$  (see [6]).  $J$  will be called the *standard* complex structure on  $(\mathcal{A}, \cdot) \bowtie (\mathcal{A}, *)$ . Setting  $\mathfrak{u} = \mathcal{A} \oplus \{0\}$ , it turns out that  $J\mathfrak{u} = \{0\} \oplus \mathcal{A}$  and both,  $\mathfrak{u}$  and  $J\mathfrak{u}$ , are abelian Lie subalgebras of  $(\mathcal{A}, \cdot) \bowtie (\mathcal{A}, *)$ .<sup>1</sup>

Consider an abelian double product  $(\mathcal{A}, \cdot) \bowtie (\mathcal{A}, *)$ . In the special case when  $*$  is the trivial product in  $\mathcal{A}$ , the corresponding Lie algebra is called an *affine Lie algebra* and is denoted by  $\mathfrak{aff}(\mathcal{A})$ . Two particular cases occur when  $(\mathcal{A}, \cdot) = \mathbb{R}$  or  $\mathbb{C}$ , obtaining in this way the Lie algebra

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<sup>1</sup> We point out that the double product  $(\mathcal{A}, \cdot) \bowtie (\mathcal{A}, *)$  is a nilpotent Lie algebra if and only if both  $(\mathcal{A}, \cdot)$  and  $(\mathcal{A}, *)$  are nilpotent commutative associative algebras [2].

of the group of affine motions of either  $\mathbb{R}$  or  $\mathbb{C}$ . The family of Lie algebras  $\mathfrak{aff}(\mathcal{A})$  where  $\mathcal{A}$  is an arbitrary associative algebra, not necessarily commutative, was considered in [8].

Condition (6) is also satisfied in the particular case when  $a * b = a \cdot b$  for all  $a, b \in \mathcal{A}$ . In this situation, we obtain:

**Lemma 3.1.** *Given a commutative associative algebra  $(\mathcal{A}, \cdot)$ , there is a holomorphic Lie algebra isomorphism  $\beta : (\mathcal{A}, \cdot) \bowtie (\mathcal{A}, \cdot) \rightarrow \mathfrak{aff}(\mathcal{A})$  with respect to the standard complex structures, given by  $\beta(a, b) = (a + b, -a + b)$ ,  $a, b \in \mathcal{A}$ .*

The next result, which is a consequence of [5, Proposition 6.1] and [6, Theorem 4.1], shows that any abelian double product is obtained as in (7) with the standard abelian complex structure defined in (8).

**Proposition 3.2.** *Let  $\mathfrak{g} = \mathfrak{u} \oplus J\mathfrak{u}$  be an abelian double product. Then the Lie bracket in  $\mathfrak{g}$  induces two structures of commutative associative algebras in  $\mathfrak{u}$ ,  $(\mathfrak{u}, \cdot)$ ,  $(\mathfrak{u}, *)$ , such that  $\mathfrak{g} = (\mathfrak{u}, \cdot) \bowtie (\mathfrak{u}, *)$ .*

*Proof.* The proposition follows by considering in  $\mathfrak{u}$  the products

$$(9) \quad x \cdot y = J([Jx, y]_{J\mathfrak{u}}), \quad x * y = [Jx, y]_{\mathfrak{u}}, \quad x, y \in \mathfrak{u}.$$

□

As a particular case of Proposition 3.2, we state the following result, which will be used below.

**Corollary 3.3.** *Let  $\mathfrak{g}$  an abelian double product of the form  $\mathfrak{g} = \mathfrak{g}' \oplus J\mathfrak{g}'$ . Then:*

- (i) *The Lie bracket in  $\mathfrak{g}$  induces a structure of commutative associative algebra on  $\mathfrak{g}'$  given by  $x * y = [Jx, y]$ ;*
- (ii) *If  $\mathcal{A}$  denotes the associative algebra  $(\mathfrak{g}', *)$  in (i), then  $\mathcal{A}^2 = \mathcal{A}$  and  $\mathfrak{g}$  is holomorphically isomorphic to  $\mathfrak{aff}(\mathcal{A})$  with its standard complex structure.*

*Proof.* (i) follows from Proposition 3.2, noting that in this case  $x \cdot y = 0$  and  $x * y = [Jx, y]$  for any  $x, y \in \mathfrak{g}'$ .

(ii) From Lemma 2.1 we have that both,  $\mathfrak{g}'$  and  $J\mathfrak{g}'$  are abelian, and therefore,  $\mathfrak{g}' = [\mathfrak{g}', J\mathfrak{g}']$ . This implies that  $\mathcal{A}^2 = \mathcal{A}$ . Setting

$$\phi : \mathfrak{aff}(\mathcal{A}) \rightarrow \mathfrak{g}, \quad \phi(x, y) = y - Jx,$$

we obtain that  $\phi$  is a holomorphic isomorphism, where  $\mathfrak{aff}(\mathcal{A})$  is equipped with its standard complex structure. □

We show next that there is a large family of Lie algebras with abelian complex structure which are not abelian double products. Let  $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{v}$  where  $\mathfrak{a} = \text{span}\{f_1, f_2\}$  and  $\mathfrak{v}$  is a  $2n$ -dimensional real vector space. We fix an endomorphism  $J$  of  $\mathfrak{g}$  such that  $J^2 = -I$ ,  $Jf_1 = f_2$  and  $\mathfrak{v}$  is  $J$ -stable. Given a linear isomorphism  $T$  of  $\mathfrak{v}$  commuting with  $J|_{\mathfrak{v}}$ , we define a Lie bracket on  $\mathfrak{g}$  such that  $\mathfrak{a}$  is an abelian subalgebra,  $\mathfrak{v}$  is an abelian ideal and the bracket between elements in  $\mathfrak{a}$  and  $\mathfrak{v}$  is given by:

$$[f_1, v] = TJ(v), \quad [f_2, v] = T(v), \quad \text{for every } v \in \mathfrak{v}.$$

It turns out that  $J$  is an abelian complex structure on  $\mathfrak{g}$  (see [8, Example 5.2]).

**Proposition 3.4.** *The Lie algebra  $\mathfrak{g}$  is not an abelian double product, unless  $n = 1$  and  $\mathfrak{g} = \mathfrak{aff}(\mathbb{C})$  with the abelian complex structure  $J_1$  from Example 1.*

*Proof.* Assume that  $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{v}$  as above is an abelian double product,  $\mathfrak{g} = \mathfrak{u} \oplus Ju$ , and let  $\{x_1, \dots, x_{n+1}\}$  be a basis of  $\mathfrak{u}$ . For  $i = 1, \dots, n+1$ , there exist  $a_i, b_i \in \mathbb{R}$ ,  $e_i \in \mathfrak{v}$  such that:

$$x_i = a_i f_1 + b_i f_2 + e_i.$$

Since  $\mathfrak{u}$  is abelian we have

$$0 = [x_i, x_k] = T((b_i I + a_i J)e_k - (b_k I + a_k J)e_i), \quad \text{for every } i, k,$$

therefore, as  $T$  is an isomorphism, it follows that

$$(10) \quad (b_i I + a_i J)e_k = (b_k I + a_k J)e_i, \quad \text{for every } i, k.$$

We note that  $a_i^2 + b_i^2 \neq 0$  for some  $i$ , otherwise we would have  $\mathfrak{u} \subset \mathfrak{v}$ , hence  $\mathfrak{g} \subset \mathfrak{v}$ , a contradiction. We may assume that  $a_1^2 + b_1^2 \neq 0$  and, by applying  $(b_1 I + a_1 J)^{-1}$  on both sides of (10) with  $i = 1$ , we obtain:

$$e_k = (b_1 I + a_1 J)^{-1}(b_k I + a_k J)e_1, \quad \text{for every } k.$$

Setting  $\mathfrak{s} = \text{span}\{f_1, f_2, e_1, J e_1\}$ , the above equation implies that  $x_k \in \mathfrak{s}$  for every  $k$ , hence,  $\mathfrak{u} \subset \mathfrak{s}$ . Since  $\mathfrak{s}$  is  $J$ -stable, it follows that  $\mathfrak{u} \oplus Ju \subset \mathfrak{s}$ , therefore,  $\mathfrak{g} = \mathfrak{s}$ . From [3, Lemma 2.8] we conclude that  $\mathfrak{s} \cong \mathfrak{aff}(\mathbb{C})$  with the abelian complex structure  $J_1$  from Example 1.  $\square$

In the following result we give a characterization of affine Lie algebras among the Lie algebras carrying an abelian complex structure. Any affine Lie algebra  $\mathfrak{g} = \mathfrak{aff}(\mathcal{A})$  can be written as  $\mathfrak{g} = \mathfrak{a} \oplus J\mathfrak{a}$  with  $\mathfrak{a}$  an abelian ideal and  $J\mathfrak{a}$  an abelian subalgebra and, moreover,  $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$ . We show next that these conditions are also sufficient.

**Theorem 3.5.** *Let  $\mathfrak{g}$  be a solvable Lie algebra with an abelian complex structure  $J$  such that  $\mathfrak{g}$  admits a vector space decomposition  $\mathfrak{g} = \mathfrak{u} + Ju$ . Then:*

- (i) *if  $\mathfrak{u}$  is an abelian subalgebra of  $\mathfrak{g}$  then  $\mathfrak{g} = \mathfrak{a} \oplus J\mathfrak{a}$  is an abelian double product with  $\mathfrak{a} \subset \mathfrak{u}$ ;*
- (ii) *if  $\mathfrak{u}$  is an abelian ideal of  $\mathfrak{g}$  and, moreover,  $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$ , then  $(\mathfrak{g}, J)$  is holomorphically isomorphic to  $\mathfrak{aff}(\mathcal{A})$  for some commutative associative algebra  $(\mathcal{A}, \cdot)$ .*

*Proof.* We may assume, in either case, that  $\mathfrak{u} \cap Ju = \mathfrak{z}$ , where  $\mathfrak{z}$  denotes the center of  $\mathfrak{g}$ . Indeed,  $\mathfrak{u} + \mathfrak{z}$  is an abelian subalgebra (respectively, abelian ideal) such that  $\mathfrak{g} = (\mathfrak{u} + \mathfrak{z}) + J(\mathfrak{u} + \mathfrak{z})$  and  $(\mathfrak{u} + \mathfrak{z}) \cap J(\mathfrak{u} + \mathfrak{z}) = \mathfrak{z}$ . To prove the last equality, let  $x + z = J(y + w)$  where  $x, y \in \mathfrak{u}$  and  $z, w \in \mathfrak{z}$ . Then  $[x + z, u] = 0$  for all  $u \in \mathfrak{u}$  and

$$[x + z, Ju] = -[J(x + z), u] = [y + w, u] = 0, \quad u \in \mathfrak{u}.$$

Since  $\mathfrak{g} = \mathfrak{u} + Ju$  and  $\mathfrak{z}$  is  $J$ -invariant, the assertion follows.

(i) Let  $\mathfrak{u} = \mathfrak{h} + \mathfrak{z}$  and  $\mathfrak{z} = \mathfrak{l} \oplus J\mathfrak{l}$ . Setting  $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{l}$ , one obtains  $\mathfrak{g} = \mathfrak{a} \oplus J\mathfrak{a}$ , with  $\mathfrak{a} \subset \mathfrak{u}$ , hence  $\mathfrak{a}$  is an abelian subalgebra, and (i) follows.

(ii) Since in this case  $\mathfrak{u}$  is an abelian ideal, one has that  $\mathfrak{g}' \subset \mathfrak{u}$ . Let  $\mathfrak{h} \subset \mathfrak{g}'$  and  $\mathfrak{k} \subset \mathfrak{u}$  be subspaces such that

$$\mathfrak{g}' = \mathfrak{h} \oplus (\mathfrak{g}' \cap \mathfrak{z}), \quad \mathfrak{u} = \mathfrak{k} \oplus (\mathfrak{g}' + \mathfrak{z}).$$

Using the condition  $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$ , we may decompose  $\mathfrak{z}$  as

$$\mathfrak{z} = (\mathfrak{g}' \cap \mathfrak{z}) \oplus \mathfrak{l} \oplus J(\mathfrak{g}' \cap \mathfrak{z}) \oplus J\mathfrak{l}.$$

Set now  $\mathfrak{v} = \mathfrak{k} \oplus \mathfrak{h} \oplus (\mathfrak{g}' \cap \mathfrak{z}) \oplus \mathfrak{l} \subset \mathfrak{u}$ . It can be verified that  $\mathfrak{v} \cap J\mathfrak{v} = \{0\}$  and  $\dim \mathfrak{v} = \frac{1}{2} \dim \mathfrak{g}$ , hence  $\mathfrak{g} = \mathfrak{v} \oplus J\mathfrak{v}$ . Since  $\mathfrak{g}' \subset \mathfrak{v} \subset \mathfrak{u}$ , it follows that  $\mathfrak{v}$  is an abelian ideal. Arguing as in the proof of Corollary 3.3, we obtain that  $\mathfrak{g} = \mathfrak{v} \oplus J\mathfrak{v}$  is holomorphically isomorphic to  $\mathfrak{aff}(\mathcal{A})$  where  $\mathcal{A}$  is the commutative associative algebra  $(\mathfrak{v}, *)$  where  $x * y = [Jx, y]$ ,  $x, y \in \mathfrak{v}$ .  $\square$

**Corollary 3.6.** *Let  $\mathfrak{g}$  be a solvable Lie algebra with an abelian complex structure  $J$ . Then:*

- (1)  $\mathfrak{g}'_J$  is an abelian double product and if  $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$ , then  $(\mathfrak{g}'_J, J)$  is holomorphically isomorphic to  $\mathfrak{aff}(\mathcal{A})$  for some commutative associative algebra  $(\mathcal{A}, \cdot)$ ;
- (2) if  $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}'$ , then  $\mathfrak{g} = \mathfrak{u} \oplus J\mathfrak{u}$  is an abelian double product for some subalgebra  $\mathfrak{u} \subset \mathfrak{g}'$ .

#### 4. HERMITIAN LIE ALGEBRAS WITH ABELIAN COMPLEX STRUCTURES

In this section we study natural questions concerning Hermitian Lie algebras with an abelian complex structure, namely, the existence of Kähler structures and the flatness of the first canonical Hermitian connection.

**4.1. Kähler Lie algebras with abelian complex structures.** We show next that such a Lie algebra is affine of a very restrictive type. More precisely, its associated simply connected Lie group is a direct product of 2-dimensional hyperbolic spaces and a flat factor.

**Theorem 4.1.** *Let  $(\mathfrak{g}, J, g)$  be a Kähler Lie algebra with  $J$  an abelian complex structure. Then  $\mathfrak{g}$  is isomorphic to*

$$\mathfrak{aff}(\mathbb{R}) \times \cdots \times \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}^{2s},$$

*and this decomposition is orthogonal and  $J$ -stable.*

*Proof.* Combining the Kähler condition  $d\omega = 0$  with (4) we obtain

$$0 = d\omega(Jx, Jy, Jz) = -\omega([Jx, Jy], Jz) - \omega([Jy, Jz], Jx) - \omega([Jz, Jx], Jy).$$

The fact that  $J$  is abelian gives

$$(11) \quad g([x, y], z) + g([y, z], x) + g([z, x], y) = 0$$

for any  $x, y, z \in \mathfrak{g}$ , and from this expression the following statements are easily verified:

- (i)  $\mathfrak{z} \subseteq (\mathfrak{g}'_J)^\perp$ .
- (ii)  $(\mathfrak{g}')^\perp$  is abelian.
- (iii)  $\text{ad}_z|_{\mathfrak{g}'}$  is symmetric for all  $z \in \mathfrak{g}$ .

We prove next that we have an equality in (i). Let  $x \in (\mathfrak{g}'_J)^\perp$ , then  $J \text{ad}_x = -\text{ad}_x J$ . Indeed, we compute

$$\begin{aligned} g(J[x, y], z) &= -g([x, y], Jz) \\ &= g([y, Jz], x) + g([Jz, x], y) \\ &= g([Jz, x], y) = -g([z, Jx], y) \\ &= g([Jx, y], z) + g([y, z], Jx) \\ &= g([Jx, y], z) \\ &= -g([x, Jy], z) \end{aligned}$$

and the claim follows. Since  $(\mathfrak{g}'_J)^\perp \subset (\mathfrak{g}')^\perp$ , it follows from (ii) that  $(\mathfrak{g}'_J)^\perp$  is abelian. Furthermore, it is  $J$ -stable and therefore we have

$$(\text{ad}_x)^2 J = -\text{ad}_x \text{ad}_{Jx} = -\text{ad}_{Jx} \text{ad}_x = \text{ad}_x J \text{ad}_x = -(\text{ad}_x)^2 J$$

which implies  $(\text{ad}_x)^2 = 0$ . Since  $\text{ad}_x$  is also symmetric (which follows easily from (11)), we obtain that  $\text{ad}_x = 0$ , that is,  $x \in \mathfrak{z}$  and therefore

$$(12) \quad \mathfrak{z} = (\mathfrak{g}'_J)^\perp \quad \text{and} \quad \mathfrak{g} = (\mathfrak{g}'_J) \oplus \mathfrak{z}.$$

It follows from Lemma 2.1(v) that  $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$ . Hence,

$$(13) \quad \mathfrak{g} = (\mathfrak{g}' \oplus J\mathfrak{g}') \oplus \mathfrak{z}.$$

Now, from Corollary 3.3 applied to  $\mathfrak{g}' \oplus J\mathfrak{g}'$ , we obtain that there is a structure of commutative associative algebra on  $\mathfrak{g}'$  given by

$$x * y = [Jx, y], \quad \text{for } x, y \in \mathfrak{g}'.$$

We denote  $\mathcal{A} = (\mathfrak{g}', *)$  and  $\ell_x : \mathfrak{g}' \rightarrow \mathfrak{g}'$  the multiplication by  $x \in \mathfrak{g}'$ . Using that  $J$  is abelian and condition (iii) above, we obtain that  $\ell_x$  is symmetric with respect to  $g|_{\mathcal{A} \times \mathcal{A}}$  for any  $x \in \mathcal{A}$ . If  $x_0 \in \mathcal{A}$  is a nilpotent element, then  $\ell_{x_0}$  is nilpotent. Since  $\ell_{x_0}$  is also symmetric, we have that  $\ell_{x_0} = 0$ , that is,  $x_0 \in \mathfrak{z}$ . It follows from (13) that  $x_0 = 0$ . Thus,  $\mathcal{A}$  has no nilpotent elements, and therefore it is semisimple. From structure theory of associative commutative algebras, we have that  $\mathcal{A}$  is a direct sum of copies of  $\mathbb{R}$  and  $\mathbb{C}$  with their canonical product. Moreover,  $\mathbb{C}$  cannot occur in the decomposition since  $\ell_x$  has real eigenvalues for any  $x \in \mathfrak{g}'$ . We conclude that

$$\mathcal{A} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n, \quad \text{with} \quad e_i^2 = e_i, \quad e_i e_j = 0, \quad i \neq j.$$

This gives a  $J$ -stable splitting

$$\mathfrak{g}' \oplus J\mathfrak{g}' = \mathfrak{aff}(\mathbb{R}e_1) \times \cdots \times \mathfrak{aff}(\mathbb{R}e_n).$$

Moreover, using that  $e_i^2 = e_i$ ,  $e_i e_j = 0$ ,  $i \neq j$  for  $i, j = 1, \dots, n$ , we obtain that the splitting above is orthogonal.  $\square$

**Corollary 4.2.** *Let  $G$  be a simply connected Lie group equipped with a left invariant Kähler structure  $(J, g)$  such that  $J$  is abelian. If the commutator subgroup is  $n$ -dimensional and the center is  $2s$ -dimensional, then*

$$G = H^2(-c_1) \times \cdots \times H^2(-c_n) \times \mathbb{R}^{2s},$$

where  $c_i > 0$ ,  $i = 1, \dots, n$ , and  $H^2(-c_i)$  denotes the 2-dimensional hyperbolic space of constant curvature  $-c_i$ .

*Proof.* Let  $e_i \in \mathfrak{g}'$  be given as in the proof of Theorem 4.1, and let  $r_i = \|e_i\| = \|Je_i\|$ . Then the simply connected Lie group with Lie algebra  $\mathfrak{aff}(\mathbb{R}e_i)$  is isometric and bilohomorphic to  $H^2(-c_i)$ , where  $c_i = \frac{1}{r_i^2}$ .  $\square$

If  $G$  is a simply connected Lie group and  $\Gamma \subset G$  is a discrete subgroup then any left invariant Hermitian structure on  $G$  gives rise to a unique Hermitian structure on  $\Gamma \backslash G$  such that the projection  $G \rightarrow \Gamma \backslash G$  is a holomorphic local isometry. This structure on  $\Gamma \backslash G$  will also be called left invariant.

**Corollary 4.3.** *Let  $M = \Gamma \backslash G$  be a compact quotient with a left invariant Kähler structure  $(J, g)$  such that  $J$  is abelian. Then  $M$  is diffeomorphic to a torus.*

*Proof.* As  $G$  admits a compact quotient, then  $G$  is a unimodular Lie group. Using the characterization given in Theorem 4.1, together with the fact that  $\mathfrak{aff}(\mathbb{R})$  is not unimodular, the corollary follows.  $\square$

**Remark 4.4.** We point out that the previous corollary holds without the assumption of the left invariance of the metric  $g$ . Indeed, following [11] (see also [24]) any Kähler metric with respect to a left invariant complex structure gives rise to a left invariant Kähler metric with respect to the same complex structure.



**Remark 4.5.** In [7] the pseudo-Riemannian geometry of abelian para-Kähler Lie algebras is studied. In particular, the authors provide conditions to ensure the flatness or Ricci-flatness of the para-Kähler (neutral) metric.

**4.2. The first canonical Hermitian connection.** Given a Hermitian Lie algebra  $(\mathfrak{g}, J, g)$ , consider the connection  $\nabla^1$  on  $\mathfrak{g}$  defined by

$$g(\nabla_x^1 y, z) = g(\nabla_x^g y, z) + \frac{1}{4}(d\omega(x, Jy, z) + d\omega(x, y, Jz)),$$

where  $\omega$  is the Kähler form. This connection satisfies

$$\nabla^1 g = 0, \quad \nabla^1 J = 0, \quad T^1 \text{ is of type } (1, 1).$$

The connection  $\nabla^1$  is known as the first canonical Hermitian connection associated to the Hermitian Lie algebra  $(\mathfrak{g}, J, g)$  (see [20, 17]). It is proved in [1, p. 21] that another expression for  $\nabla^1$  in terms of the Levi-Civita connection  $\nabla^g$  is given by

$$(14) \quad \nabla_x^1 y := \nabla_x^g y + \frac{1}{2}(\nabla_x^g J)Jy = \frac{1}{2}(\nabla_x^g y - J\nabla_x^g Jy),$$

for  $x, y \in \mathfrak{g}$ . More generally, if  $\nabla$  is any connection on  $\mathfrak{g}$ , define

$$(15) \quad \overline{\nabla}_x y := \nabla_x y + \frac{1}{2}(\nabla_x J)Jy = \frac{1}{2}(\nabla_x y - J\nabla_x Jy),$$

for  $x, y \in \mathfrak{g}$ . It is easy to see that  $\overline{\nabla}J = 0$  and, furthermore, if  $\nabla$  is torsion-free, then  $\overline{T}(x, y) = \overline{T}(Jx, Jy)$ , i.e.  $\overline{T}$  is of type  $(1, 1)$  with respect to  $J$ .

We prove in the next result that when  $\nabla$  is torsion-free and  $\overline{\nabla}$  coincides with the  $(-)$ -connection, then the complex structure  $J$  is abelian, and therefore, the Lie algebra is 2-step solvable.

**Lemma 4.6.** *Let  $\nabla$  be a torsion-free connection and  $J$  a complex structure on  $\mathfrak{g}$ . Assume that  $\overline{\nabla} = 0$ , that is,  $\nabla_x J = -J\nabla_x$  for every  $x \in \mathfrak{g}$ . Then  $J$  is abelian.*

*Proof.* If  $\overline{\nabla} = 0$ , then  $\overline{T}(x, y) = -[x, y]$ , where  $\overline{T}$  denotes the torsion of  $\overline{\nabla}$ . Since  $\overline{T}$  is of type  $(1, 1)$ , it follows that  $J$  is abelian.  $\square$

If in Lemma 4.6 the connection  $\nabla$  is the Levi-Civita connection of a Hermitian metric on  $(\mathfrak{g}, J)$ , then a much stronger restriction occurs, namely,  $\mathfrak{g}$  is abelian. This is the content of the next theorem.

**Theorem 4.7.** *Let  $(\mathfrak{g}, J, g)$  be a Hermitian Lie algebra such that its associated first canonical connection  $\nabla^1$  satisfies  $\nabla_x^1 y = 0$  for every  $x, y \in \mathfrak{g}$ , that is,  $\nabla^1$  coincides with the  $(-)$ -connection. Then  $\mathfrak{g}$  is abelian.*

*Proof.* Let  $\nabla$  be the Levi-Civita connection of  $g$ . From  $\nabla^1 \equiv 0$  and equation (15), it follows that  $\nabla_x J = -J\nabla_x$  for every  $x \in \mathfrak{g}$ , therefore

$$(16) \quad g(\nabla_x Jy, z) = -g(J\nabla_x y, z) = g(\nabla_x y, Jz),$$

for every  $x, y, z \in \mathfrak{g}$ . Moreover, Lemma 4.6 implies that  $J$  is abelian. Using equation (3) for the first and third terms in (16) and the fact that  $J$  is abelian, we get:

$$(17) \quad g([x, Jy], z) - g([x, y], Jz) + 2g([y, Jz], x) + g([z, x], Jy) - g([Jz, x], y) = 0.$$

Computing the sum of (17) over cyclic permutations of  $x, y, z$  we obtain the following equation:

$$(18) \quad g([x, Jy], z) + g([y, Jz], x) + g([z, Jx], y) = 0,$$

for every  $x, y, z \in \mathfrak{g}$ . The following conditions are consequences of (18):

- (i)  $\mathfrak{z} \subseteq (\mathfrak{g}'_J)^\perp$ .
- (ii)  $(\mathfrak{g}'_J)^\perp$  is abelian.
- (iii)  $\text{ad}_x$  is skew-symmetric for all  $x \in (\mathfrak{g}'_J)^\perp$ .

We prove next that we have an equality in (i). Let  $x \in (\mathfrak{g}'_J)^\perp$ , then  $J \text{ad}_x = -\text{ad}_x J$ . Indeed, since  $J$  is abelian, we have  $\text{ad}_x J = -\text{ad}_{Jx}$ , and therefore it follows from (iii) that both  $\text{ad}_x$  and  $\text{ad}_x J$  are skew-symmetric. By taking the adjoint of  $(\text{ad}_x J)$ , the claim easily follows. Since  $(\mathfrak{g}'_J)^\perp$  is abelian and  $J$ -stable (see (ii)), we have

$$(\text{ad}_x)^2 J = -\text{ad}_x \text{ad}_{Jx} = -\text{ad}_{Jx} \text{ad}_x = \text{ad}_x J \text{ad}_x = -(\text{ad}_x)^2 J$$

which implies  $(\text{ad}_x)^2 = 0$ . From (iii) above we obtain  $\text{ad}_x = 0$ , that is,  $x \in \mathfrak{z}$ , and therefore

$$(19) \quad \mathfrak{z} = (\mathfrak{g}'_J)^\perp \quad \text{and} \quad \mathfrak{g} = (\mathfrak{g}'_J) \oplus \mathfrak{z}.$$

It follows from Lemma 2.1(v) that  $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$ . Hence,

$$(20) \quad \mathfrak{g} = (\mathfrak{g}' \oplus J\mathfrak{g}') \oplus \mathfrak{z}.$$

Now, from Corollary 3.3 applied to  $\mathfrak{g}' \oplus J\mathfrak{g}'$ , we obtain that  $\mathfrak{g} = \mathfrak{aff}(\mathcal{A}) \oplus \mathfrak{z}$ , with  $\mathcal{A}^2 = \mathcal{A}$ . Furthermore, equation (18) can be read as

$$(21) \quad g(x * y, z) + g(y * z, x) + g(z * x, y) = 0, \quad x, y, z \in \mathfrak{g}'.$$

Let us suppose that  $\mathcal{A} \neq \{0\}$ , so that  $\mathcal{A}$  is not nilpotent. There exists an idempotent element  $e \in \mathcal{A}$ ,  $e \neq 0$ ,  $e^2 = e$ . Setting  $x = y = z = e$  in (21), one obtains  $g(e, e) = 0$ , a contradiction. Therefore  $\mathcal{A} = \{0\}$  and this means that  $\mathfrak{g}$  is abelian.  $\square$

**Remark 4.8.** A similar result for the Chern connection does not hold. See [15] for results concerning the flatness of the Chern connection on nilmanifolds.

**Remark 4.9.** In contrast with Theorem 4.7, we point out that every abelian double product can be endowed with a torsion-free connection  $\nabla$  such that  $\overline{\nabla} = 0$ .

If  $G$  is a simply connected Lie group with Lie algebra  $\mathfrak{g}$  and  $\Gamma \subset G$  is any discrete subgroup of  $G$  then the  $(-)$ -connection on  $\mathfrak{g}$  induces a unique connection on  $\Gamma \backslash G$  such that the parallel vector fields are  $\pi$ -related with the left invariant vector fields on  $G$ , where  $\pi : G \rightarrow \Gamma \backslash G$  is the projection. This induced connection on  $\Gamma \backslash G$ , which will be denoted  $\nabla^0$ , is complete, has trivial holonomy, its torsion is parallel and  $\pi$  is affine.

As a consequence of Theorem 4.7, we obtain

**Corollary 4.10.** *Let  $M = \Gamma \backslash G$  be a compact quotient of a simply connected Lie group  $G$  by a discrete subgroup  $\Gamma$ . If  $(J, g)$  is a left invariant Hermitian structure on  $M$  such that its first canonical connection  $\nabla^1$  coincides with the connection  $\nabla^0$ , then  $M$  is diffeomorphic to a torus.*

Motivated by Theorem 4.7, we study next the more general case of Hermitian structures on Lie algebras whose associated first canonical connection is flat, when the complex structure is abelian. The following lemma will be useful to prove some of the next results.

**Lemma 4.11.** *Let  $(\mathfrak{g}, J, g)$  be a Hermitian Lie algebra with  $J$  abelian. If the associated first canonical connection  $\nabla^1$  is flat, then  $\mathfrak{z} \cap \mathfrak{g}' = \{0\}$ .*

*Proof.* It follows from equations (14) and (3) and the fact that  $J$  is abelian that:

(22)

$$g(\nabla_x^1 y, z) = \frac{1}{4} (g([x, y], z) + g([z, x], y) + g([x, Jy], Jz) + g([Jz, x], Jy) - 2g([y, z], x)),$$

for every  $x, y, z \in \mathfrak{g}$ . If  $x \in \mathfrak{z} \cap \mathfrak{g}'$ , using Lemma 2.2, equation (22) becomes

$$0 = g(\nabla_x^1 y, z) = -\frac{1}{2}g([y, z], x),$$

therefore,  $x \in (\mathfrak{g}')^\perp$  which implies  $x = 0$ .  $\square$

As a straightforward consequence of Lemma 4.11, we show next that a nilpotent Lie algebra with an abelian complex structure admits no Hermitian metric with flat first canonical connection. In Theorem 4.14 below we extend this result to an arbitrary Lie algebra carrying an abelian complex structure.

**Proposition 4.12.** *Let  $(\mathfrak{g}, J, g)$  be a Hermitian Lie algebra with  $\mathfrak{g}$  nilpotent and  $J$  abelian. If the associated first canonical connection  $\nabla^1$  is flat, then  $\mathfrak{g}$  is abelian.*

*Proof.* Consider the descending central series of  $\mathfrak{g}$  defined by  $\mathfrak{g}^0 = \mathfrak{g}$ ,  $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$ ,  $i \geq 1$ . The Lie algebra  $\mathfrak{g}$  is  $k$ -step nilpotent if  $\mathfrak{g}^k = \{0\}$  and  $\mathfrak{g}^{k-1} \neq \{0\}$ .

Let us suppose that  $\mathfrak{g}$  is  $k$ -step nilpotent with  $k \geq 2$ . Then  $\mathfrak{g}^{k-1} \subset \mathfrak{z} \cap \mathfrak{g}'$  and it follows from Lemma 4.11 that  $\mathfrak{g}^{k-1} = \{0\}$ , a contradiction. Therefore,  $k = 1$  and  $\mathfrak{g}$  is abelian.  $\square$

**Corollary 4.13.** *Let  $N = \Gamma \backslash G$  be a nilmanifold with a left invariant Hermitian structure  $(J, g)$  such that  $J$  is abelian. If the associated first canonical connection  $\nabla^1$  is flat, then  $N$  is diffeomorphic to a torus.*

The next result shows that the nilpotency assumption can be dropped in Proposition 4.12. In other words, there are no non-abelian Hermitian Lie algebras with abelian complex structure and flat first canonical connection.

**Theorem 4.14.** *Let  $(\mathfrak{g}, J, g)$  be a Hermitian Lie algebra with  $J$  abelian. If the associated first canonical connection  $\nabla^1$  is flat, then  $\mathfrak{g}$  is abelian.*

The proof of this theorem will follow from Lemmas 4.15 and 4.16, which are proved next.

**Lemma 4.15.** *Let  $(\mathfrak{g}, J, g)$  be a Hermitian Lie algebra with  $J$  abelian. If the associated first canonical connection  $\nabla^1$  is flat, then both,  $\mathfrak{g}'_J$  and  $(\mathfrak{g}'_J)^\perp$  are abelian. Moreover, if  $v, w \in (\mathfrak{g}'_J)^\perp$  then  $\nabla_v^1 w = 0$ .*

*Proof.* For any  $x, y, z \in \mathfrak{g}'$ , using (22) together with Lemma 2.2 and the fact that  $\mathfrak{g}'$  is abelian we obtain:

$$0 = g(\nabla_x^1 y, Jz) = \frac{1}{4} (g([Jz, x], y) - g([x, Jy], z) - 2g([y, Jz], x)),$$

or equivalently:

$$(23) \quad 2g([y, Jz], x) = -g([z, Jx], y) - g([x, Jy], z).$$

Using (23) we compute

$$2g([y, Jz], x) - 2g([z, Jx], y) = -g([z, Jx], y) + g([y, Jz], x),$$

which implies  $g([y, Jz], x) = g([z, Jx], y)$  for every  $x, y, z \in \mathfrak{g}'$ . Therefore,  $\text{ad}_{Jz} : \mathfrak{g}' \rightarrow \mathfrak{g}'$  is symmetric for all  $z \in \mathfrak{g}'$ . In particular, using that  $\text{ad}_{Jx}$  is symmetric in equation (23) we obtain

$$2g([y, Jz], x) = -2g([x, Jy], z), \quad \text{equivalently,} \quad g([Jy, z], x) = -g([Jy, x], z),$$

that is,  $\text{ad}_{Jy}$  is skew-symmetric for all  $y \in \mathfrak{g}'$ . Hence,  $\text{ad}_{Jy}|_{\mathfrak{g}'} = 0$  for all  $y \in \mathfrak{g}'$  which implies that  $\mathfrak{g}'_J$  is abelian.

Now taking  $x \in \mathfrak{g}'$ ,  $y, z \in (\mathfrak{g}'_J)^\perp$  in (22) together with Lemma 2.2 and the fact that  $(\mathfrak{g}'_J)^\perp$  is  $J$ -stable we obtain:

$$0 = g(\nabla_x^1 y, z) = -\frac{1}{2}g([y, z], x),$$

and therefore,  $(\mathfrak{g}'_J)^\perp$  is abelian.

Since  $(\mathfrak{g}'_J)^\perp$  is abelian and  $J$ -stable, it follows from (22) that  $\nabla_v^1 w = 0$  whenever  $v, w \in (\mathfrak{g}'_J)^\perp$ .  $\square$

**Lemma 4.16.** *Let  $(\mathfrak{g}, J, g)$  be a Hermitian Lie algebra with  $J$  abelian. If the associated first canonical connection  $\nabla^1$  is flat then  $\mathfrak{g}'$  is  $J$ -stable and therefore  $\mathfrak{g}'_J = \mathfrak{g}'$ .*

*Proof.* Let  $\mathfrak{v}$  denote the orthogonal complement of  $\mathfrak{g}'$  in  $\mathfrak{g}'_J$ , so that  $\mathfrak{g}'_J = \mathfrak{g}' \oplus \mathfrak{v}$ . Taking  $x \in \mathfrak{g}'$ ,  $y \in \mathfrak{v}$ ,  $w \in (\mathfrak{g}'_J)^\perp$  and using (22) together with Lemma 2.2 we obtain

$$0 = g(\nabla_x^1 y, w) = g([Jw, x], Jy) - 2g([y, w], x),$$

and therefore

$$2g([w, y], x) = -g([Jw, x], Jy).$$

In particular, if  $x = [w, y]$ , we have

$$(24) \quad 2\|[w, y]\|^2 = -g([Jw, [w, y]], Jy) = -g([w, [Jw, y]], Jy)$$

using Jacobi identity and Lemma 4.15. Replacing  $w$  by  $Jw$  in (24), we get

$$2\|[Jw, y]\|^2 = g([Jw, [w, y]], Jy) = g([w, [Jw, y]], Jy),$$

which together with (24) implies  $[w, y] = 0$ . Since  $(\mathfrak{g}'_J)^\perp$  is abelian, it follows that  $y \in \mathfrak{z}$ , hence  $\mathfrak{v} \subset \mathfrak{z}$ . Now,  $Jy \in \mathfrak{g}'_J \cap \mathfrak{z}$ , hence we can write  $Jy = x + v$ , with  $x \in \mathfrak{g}'$ ,  $v \in \mathfrak{v}$ . It follows that  $x \in \mathfrak{z}$ , and using Lemma 4.11 we obtain  $x = 0$ , so that  $\mathfrak{v}$  is  $J$ -invariant. Since  $J$  is orthogonal, we have that  $\mathfrak{g}'$  is also  $J$ -invariant, thus  $\mathfrak{v} = 0$  and  $J\mathfrak{g}' = \mathfrak{g}'$ .  $\square$

*Proof of Theorem 4.14.* It follows from Lemma 4.16 that  $\mathfrak{g}'_J = \mathfrak{g}'$ , and therefore there is an orthogonal decomposition  $\mathfrak{g} = \mathfrak{g}' \oplus (\mathfrak{g}')^\perp$ , where both subalgebras are  $J$ -stable and abelian (Lemma 4.15). Taking  $x, y \in \mathfrak{g}'$ ,  $z \in (\mathfrak{g}')^\perp$  in (22) we obtain

$$(25) \quad g([z, x], y) - g([z, Jx], Jy) + 2g([z, y], x) = 0.$$

Changing  $x$  by  $Jx$  and  $y$  by  $Jy$  in the equation above, and adding both equations we get

$$g([z, y], x) + g([z, Jy], Jx) = 0.$$

Since this holds for every  $x \in \mathfrak{g}'$  and  $\mathfrak{g}'$  is  $J$ -invariant, we obtain:

$$(26) \quad J[z, y] = -[z, Jy], \quad z \in (\mathfrak{g}')^\perp, y \in \mathfrak{g}'.$$

Using this fact in (25), it follows that

$$g([z, x], y) + g([z, y], x) = 0,$$

and therefore  $\text{ad}_z : \mathfrak{g}' \rightarrow \mathfrak{g}'$  is skew-symmetric for any  $z \in (\mathfrak{g}')^\perp$ .

Let us take now  $z \in (\mathfrak{g}')^\perp$  and  $x, y \in \mathfrak{g}'$  and compute

$$\begin{aligned} 4g(\nabla_z^1 x, y) &= g([z, x], y) + g([y, z], x) + g([z, Jx], Jy) + g([Jy, z], Jx) - 2g([x, y], z) \\ &= 2g([z, x], y) + 2g([z, Jx], Jy) \\ &= 0, \end{aligned}$$

where we used that  $\text{ad}_z$  is skew-symmetric in the first equality and (26) in the second equality. Combining this with the fact that  $\nabla_v^1 w = 0$  for  $v, w \in (\mathfrak{g}')^\perp$  (Lemma 4.15), we obtain that  $\nabla_z^1 = 0$  for any  $z \in (\mathfrak{g}')^\perp$ . Since  $\nabla_x^1 = 0$  for any  $x \in \mathfrak{g}'$  (Lemma 2.2), it follows that  $\nabla^1 = 0$ , and therefore  $\mathfrak{g}$  is abelian, according to Theorem 4.7.  $\square$

**Remark 4.17.** According to Theorem 4.14, there are no non-abelian Hermitian Lie algebras with abelian complex structure and flat first canonical connection. On the other hand, we note that there exist Hermitian Lie algebras with flat first canonical connection and non-abelian complex structure. Indeed, in [9] it was shown that any even-dimensional Lie algebra with a flat metric  $g$  admits a compatible complex structure  $J$  such that  $(J, g)$  is Kähler, following results of Milnor in [22]. As in this case we have that  $\nabla^1 = \nabla^g$ , it follows that  $\nabla^1$  is flat. Moreover,  $J$  is not abelian, unless the Lie algebra is abelian (see Theorem 4.1). It would be interesting to find examples of flat first canonical connections in the non-Kähler case.

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